Journal of Geometry and Physics 39 (2001) 62–80

On the cohomology of a polarization which contains the generators of a free torus action

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Received 12 April 2000

Abstract

In [Trans. Am. Math. Soc. 230 (1977) 235], Rawnsley studies a strongly integrable subtangent bundle F on a smooth manifold M which contains the generators of a free circle action on M and a line bundle $L \to M$ with a flat F-connection ∇. It is proved that the cohomology groups of the sheaf $\mathcal{S}_F$ of sections of L which are covariantly constant along F can be injectively mapped in the cohomology groups of the restriction of $\mathcal{S}_F$ on the Bohr–Sommerfeld set Y of the action on M. In the present paper, we discuss similar results for free torus actions as well as some consequences in the context of the geometric quantization of a symplectic manifold with a Hamiltonian action of a torus T^k. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 53C99; 81S10; 46M20

Subj. Class.: Differential geometry; Quantum mechanics

Keywords: Sheaf cohomology; Geometric quantization; Polarization; Bohr–Sommerfeld set

1. Introduction

Let M be a smooth manifold and F a strongly integrable sub-bundle of $TM^{\mathbb{C}}$. Let $L \to M$ be a line bundle with a flat, Hermitian F-connection ∇ and $\mathcal{S}_F$ be the sheaf of germs of local sections of L, which are covariantly constant along F. Then the cohomology $H^*(M, \mathcal{S}_F)$ can be identified with the de Rham cohomology of a complex of differential forms on F with values in L. Assume that the circle T^1 acts freely on M and that all infinitesimal generators are sections of F. The *Bohr–Sommerfeld set Y of the action* is the subset of M consisting of all points with the property that ∇ has trivial holonomy along the orbit of each of these points. Therefore Y can be given as the inverse image $l^{-1}(1)$, where l is the function $l : M \to T^1$, which measures the holonomy along the orbits of T^1. If 1 is a regular value of l then Y is a smooth submanifold of M with the property that F is tangent to Y. Furthermore, the line bundle with the connection restricts to Y and the cohomology groups

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PII: S0393-0440(00)00075-9

of the restriction of $\mathcal{S}_F$ on Y can be calculated using the same de Rham theorem. In [8], a map $\hat{J} : H^p(M, \mathcal{S}_F) \to H^{p-1}(Y, \mathcal{S}_F)$ is defined for $p \geq 1$ by averaging parallel transport of differential forms and then contracting by an infinitesimal generator of the action which has period one. It is then proved (Theorem 5 in [8]) that $H^p(M, \mathcal{S}_F) = \{0\}$ and that $\hat{J}$ is an isomorphism for $p = 1$ and injective for $p > 1$. Moreover, if $Z = Y/T^1$ then the push-forward of F under the orbit map defines a strongly integrable sub-bundle $\tilde{F}$ and using parallel transport on L along the orbits we can define a quotient bundle on Z with an induced flat $\tilde{F}$-connection. Then a new map $\tilde{J} : H^p(M, \mathcal{S}_F) \to H^{p-1}(Z, \mathcal{S}_{\tilde{F}})$ can be defined which turns out to be an isomorphism for $p = 1$ and injective for $p > 1$.

In this paper, we make a similar construction for a free action of a k-dimensional torus T^k on M assuming that all the infinitesimal generators of the action are sections of F. We define the Bohr–Sommerfeld set of the action as the subset Y of M which consists of those T^k-orbits. Then, using the ideas in [8], we define new maps $\hat{J} : H^p(M, \mathcal{S}_F) \to H^{p-k}(Y, \mathcal{S}_F)$ and $\tilde{J} : H^p(M, \mathcal{S}_F) \to H^{p-k}(Z, \mathcal{S}_{\tilde{F}})$ for $p \geq k$. Theorems 4.2 and 4.4 state that the maps $\hat{J}$, $\tilde{J}$ have similar properties with their analogues in the case that $k = 1$, and also Theorem 5.3 for the case of a symplectic manifold.

Although the above construction does not rely on any symplectic structure on M, it yields interesting results in the context of geometric quantization of a symplectic manifold. Let (M, ω) be an integral symplectic manifold, (L, ∇) prequantization data and F a polarization which contains the generators of a free, Hamiltonian action of the circle T^1 on M. One approach to geometric quantization is to consider the cohomology groups $H^*(M, \mathcal{S}_F)$ together with a Poisson subalgebra of $C^\infty(M)$ which can be "quantized" to give a Lie algebra of operators on $H^*(M, \mathcal{S}_F)$ as the quantization of (M, ω). The action of T^1 lifts to an action on L by bundle automorphisms and therefore acts on the sheaf $\mathcal{S}_F$ and on its cohomology. The space Z is the union of reduced spaces at integral levels of the Hamiltonian, which generates the action and $H^*(Z, \mathcal{S}_{\tilde{F}})$ is the quantization of these reduced spaces. Therefore the theory developed in [8] relates the quantization of (M, ω) with the quantizations of all the integral reduced spaces.

The question of relating symplectic reduction and quantization has been studied extensively in the last 20 years for Hamiltonian actions of compact, connected Lie groups. For the definition of quantization either sheaf cohomology has been used [3,9] or the index of a Spin$^{\mathrm{c}}$–Dirac operator, which is defined with respect to a positive almost complex structure [2,6,7,11,12]. In the sheaf cohomological context the question has been mainly studied for strictly positive polarizations, i.e. for Kähler manifolds in [3,9]. If for $p > 0$ the cohomology groups $H^p(M, \mathcal{O}(L))$, vanish and the same is true for the corresponding cohomology groups of the induced sheaves on the reduced spaces at the integral levels of the momentum map, then one expects to build $H^0(M, \mathcal{O}(L))$ from the spaces $H^0(M_f, \mathcal{O}(L_f))$ for some elements f in the image of the momentum map in $\mathfrak{g}^*$. It seems therefore interesting to study these questions using a non-strictly positive polarization. This is what our construction does in the case that F has at least k real directions, which are tangent to the orbits of a free action of T^k. Our result in this direction is Theorem 5.3.

In Sections 2 and 3, we explain the theory developed in [8] with some extensions which allow us to generalize the construction to the case of a free action of a torus T^k, the generators of which are all in F. In particular, Theorem 3.4 of Section 3 is an improved version of Theorem 5 in [8]. In Section 4, we generalize the constructions in [8] in the case of a free action of a torus T^k with its generators in F. In particular, Theorems 4.2 and 4.4 are the analogues of Theorems 3.4 and 3.8 in [8]. Finally, in Section 5, we apply the theory to the geometric quantization of a symplectic manifold where we relate our results to the question of whether symplectic reduction and geometric quantization commute (Theorem 5.3).

2. A de Rham theorem for line bundle valued forms

Let M be a smooth manifold and $p : E \to M$ a smooth vector bundle over M. We shall denote by $\Gamma(E)$ the space of smooth global sections of E. If E is the tangent bundle TM we also use the notation $\mathcal{U}(M)$ for the space $\Gamma(TM)$ and $\Omega^p(M)$ for $\Gamma(\wedge^p T^*M)$. If U is open in M, we denote by $\mathcal{U}(U)$ the space $\Gamma(TM|_U)$ and by $\Omega^p(U)$ the space $\Gamma(\wedge^p T^*M|_U)$.

A *strongly integrable subtangent bundle* F on M is a smooth sub-bundle of the complexification $TM^{\mathbb{C}}$ of TM which has the following properties:

1. F is involutive,
2. $F + \bar{F}$ is also involutive,
3. $\dim_{\mathbb{C}} F_x \cap \bar{F}_x$ is constant for all $x \in M$.

Let $C_F(M)$ be the subset of $C^\infty(M)$ consisting of those smooth functions ϕ, with the property that $\xi\phi = 0$ for all vector fields ξ in $\Gamma(F)$. Let $\Omega_F^p(M)$ be the space $\Gamma(\wedge^p F^*)$. We denote by

$$d^F : \Omega_F^p(U) \to \Omega_F^{p+1}(U)$$

by setting

$$\begin{aligned} d^F\alpha(\xi_1, \dots, \xi_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \alpha(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_{p+1}) \\ &\quad + \sum_{i<j} \alpha([\xi_i, \xi_j], \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{p+1}) \end{aligned}$$

for $\xi_i \in \mathcal{U}_F(U)$, $i = 1, \dots, p+1$, $\alpha \in \Omega_F^p(U)$. Clearly, $d^F \circ d^F = 0$. The spaces $\Omega_F^p(U)$, U open in M for each p, form a presheaf. Let $\mathcal{D}_F^p$ be the associated sheaf. This is the sheaf of germs of local sections of $\wedge^p F^*$. Then, for each p, d^F induces a map of sheaves $d^F : \mathcal{D}_F^p \to \mathcal{D}_F^{p+1}$.

An infinitesimal automorphism of a subtangent bundle F is a vector field ξ in $\mathcal{U}(M)$, such that for all η in $\mathcal{U}_F(M)$, the Lie bracket $[\xi, \eta]$ lies in $\mathcal{U}_F(M)$. Obviously, if F is involutive then every vector field in $\mathcal{U}_F(M)$ is an infinitesimal automorphism of F.

Let ξ be an infinitesimal automorphism of F and let $\alpha \in \Omega_F^p(M)$. We define the Lie derivative $\mathcal{L}_\xi\alpha$ of α in the direction of X by setting

$$\mathcal{L}_\xi\alpha(\xi_1, \dots, \xi_p) = \xi\alpha(\xi_1, \dots, \xi_p) - \sum_{i=1}^{p} \alpha(\xi_1, \dots, [\xi, \xi_i], \dots, \xi_p) \tag{1}$$

for all $\xi_1, \dots, \xi_p$ in $\mathcal{U}_F(M)$. As with ordinary forms in $\Omega^p(M)$, we have Cartan's identity for elements in $\Omega_F^p(M)$, which reads as

$$\mathcal{L}_\xi = d^F \circ i(\xi) + i(\xi) \circ d^F,$$

where $i(\xi) : \Omega_F^p(L) \to \Omega_F^{p-1}(L)$ is the map given by contraction with the vector field ξ. Suppose that the vector field $\xi \in \mathcal{U}_F(M)$ generates the one parameter flow σ_t, $t \in \mathbb{R}$. Then, in terms of this flow the Lie derivative $\mathcal{L}_\xi$ is given by

$$\mathcal{L}_\xi \alpha = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_{-t}{}^* \alpha.$$

For more details, we refer to [1].

Let $L \to M$ be a line bundle over M and F a subtangent bundle on M. A *partial F-connection* on L is a linear map

$$\nabla : \Gamma(L) \to \Gamma(L \otimes F^*),$$

which satisfies the Leibniz rule, i.e. $\forall \phi \in C^\infty(M)$ and $\forall s \in \Gamma(L)$,

$$\nabla(\phi s) = \phi \nabla s + d^F \phi \otimes s.$$

If ξ is a vector field in $\mathcal{U}_F(M)$, we use the usual notation $\nabla_\xi s$ for $(\nabla s)(\xi)$. We also write $S_F^0(L)$ instead of $\Gamma(L)$ and similarly we denote by $S_F^p(L)$ the space $\Gamma(L \otimes \wedge^p F^*)$ for $p = 1, 2, \dots$. Since F is involutive, we can use the F-connection ∇ to define maps

$$\partial^F : S_F^p(L) \to S_F^{p+1}(L)$$

by setting

$$\begin{aligned}\partial^F \alpha(\xi_1, \dots, \xi_{p+1}) = {} & \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{\xi_i} \alpha(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_{p+1}) \\ & \times \sum_{i<j} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{p+1}) \end{aligned} \tag{2}$$

for all $\alpha \in S_F^p(L)$ and $\xi_i \in \mathcal{U}_F(M)$, $i = 1, \dots, p$, where with the notation $\hat{\xi}_1$, we mean that we omit ξ_i. It is straightforward to check that for $\alpha \in S_F^p(L)$,

$$\partial^F \circ \partial^F \alpha = \omega \wedge \alpha,$$

where ω is the curvature of ∇, i.e. $\omega \in \Lambda^2 F^*$ and satisfies the equation

$$([\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]})s = \omega(\xi, \eta)s$$

for all $\xi, \eta \in \mathcal{U}_F(M)$. It follows that $\partial^F \circ \partial^F = 0$ if and only if $\omega = 0$, i.e. if and only if ∇ is flat. In what follows we shall often use ∂^F to denote ∇ for $p = 0$.

Assume now that F is a strongly integrable subtangent bundle on M and ∇ is a flat F-connection on the line bundle $L \to M$. For an open set U in M, we consider the spaces

$S_F^p(L|_U)$, and the maps $\partial^F : S_F^p(L|_U) \to S_F^{p+1}(L|_U)$ for U open in M, $p = 0, 1, \dots$. These spaces form a presheaf for each p and ∂^F maps of presheaves. We denote by $\mathcal{S}_F^p(L)$ the associated sheaf (for each p) and by $\partial^F : \mathcal{S}_F^p(L) \to \mathcal{S}_F^{p+1}(L)$ the associated map of sheaves. In particular, $\mathcal{S}_F^0(L)$ is the sheaf of germs of local sections of L. Obviously, $\mathcal{S}_F^p(L)$ is isomorphic to $S_F^0(L) \otimes \mathcal{D}_F^p$. Let $\mathcal{S}_F$ be the sheaf of those local sections, which are constant along F-directions. Then $\mathcal{S}_F = \mathrm{Ker}(\partial^F : \mathcal{S}_F^0(L) \to \mathcal{S}_F^1(L))$. Let S_F be the space of global sections of $\mathcal{S}_F$. Obviously, $S_F = \mathrm{Ker}(\partial^F : S_F^0(L) \to S_F^1(L))$. We state a de Rham theorem which calculates the cohomology of $\mathcal{S}_F$ in terms of the forms $S_F^p(L)$.

Theorem 2.1 (Rawnsley [8]). *Let F be an involutive, strongly integrable subtangent bundle on M, $L \to M$ a line bundle over M and ∇ a flat F-connection on L. Then the sequence*

$$0 \to \mathcal{S}_F \hookrightarrow \mathcal{S}_F^0(L) \xrightarrow{\partial^F} \mathcal{S}_F^1(L) \xrightarrow{\partial^F} \cdots \xrightarrow{\partial^F} \mathcal{S}_F^n \xrightarrow{\partial^F} 0 \tag{3}$$

is a fine resolution of $\mathcal{S}_F$. This gives rise to isomorphisms between $H^i(M, \mathcal{S}_F)$ and the corresponding cohomology groups of the complex of L-valued differential forms

$$0 \to S_F^0(L) \xrightarrow{\nabla} S_F^1(L) \xrightarrow{\partial^F} \cdots \xrightarrow{\partial^F} S_F^n(L) \xrightarrow{\partial^F} 0. \tag{4}$$

The proof is a standard argument proving that a long sequence of sheaves is exact. For the complete argument, we refer to the proofs of Theorem 3 and Corollary 2 in [8]. Theorem 2.1 is quite important for our discussion, because we shall define maps between cohomology groups by defining maps between closed differential forms with values in L.

Let ξ be an infinitesimal automorphism of F and let $\alpha \in S_F^p(L)$. We define the Lie derivative $\mathcal{L}_\xi \alpha$ of α in the direction of ξ by setting

$$\mathcal{L}_\xi \alpha(\xi_1, \dots, \xi_p) = \nabla_\xi \alpha(\xi_1, \dots, \xi_p) - \sum_{i=1}^{p} \alpha(\xi_1, \dots, [X, \xi_i], \dots, \xi_p) \tag{5}$$

for all $\xi_1, \dots, \xi_p$ in $\mathcal{U}_F(M)$. Cartan's identity for differential forms in $\Omega_F^p(L)$ now becomes

$$\mathcal{L}_\xi = \partial^F \circ i(\xi) + i(\xi) \circ \partial^F,$$

where $i(\xi) : S_F^p(L) \to S_F^{p-1}(L)$ is the map given by contraction with the vector field ξ. Assume, as before, that the vector field $\xi \in \mathcal{U}_F(M)$ generates the one parameter flow σ_t, $t \in \mathbb{R}$. It turns out again that we can express the Lie derivative $\mathcal{L}_\xi$ in terms of this flow as follows. For $x \in M$ and $t \in \mathbb{R}$, we define the map $P_{x,t} : L_{\sigma_t(x)} \to L_x$ given by parallel transport along the curve $\gamma_{x,t} : [0, t] \to M$, where for $s \in [0, t]$, $\gamma_{x,t}(s) = \sigma_{t-s}(x)$. We can now define the map

$$\Sigma_t : S_F^p(L) \to S_F^p(L)$$

by setting

$$(\Sigma_t \alpha)_x(\xi_1, \dots, \xi_p) = P_{x,t}(\alpha_{\sigma_t(x)}(\sigma_{t*}\xi_1, \dots, \sigma_{t*}\xi_p)) \tag{6}$$

for $x \in M, \xi_1, \ldots, \xi_p \in F_x$. It is straightforward to check that for $t, s \in \mathbb{R}$

$$\Sigma_{t+s} = \Sigma_t \circ \Sigma_s. \tag{7}$$

Assuming that t is small enough, we can write α in local coordinates and use the formula for the parallel transport in Lemma 1.9.1 in [4]. Then one can prove that

$$\mathcal{L}_\xi \alpha = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Sigma_{-t}\alpha. \tag{8}$$

3. The Bohr–Sommerfeld set of a periodic flow with generator in a subtangent bundle

Let F be an involutive, strongly integrable subtangent bundle on a smooth manifold M and let $L \to M$ be a line bundle over M with a flat F-connection ∇. Assume also that there is a Hermitian structure on L which is compatible with ∇. Let ξ be a vector field in $\mathcal{U}_F(M)$ and assume that it generates a periodic flow σ_t, $t \in \mathbb{R}$ with period one, i.e. $\sigma_t = \sigma_{1+t}$ for all $t \in \mathbb{R}$. We assume that the flow does not have any fixed points. Alternatively, we can think of this flow as a free smooth action τ of the circle T^1 on M, with ξ as an infinitesimal generator of the action, which lies in F and its flow has period one. Let $\hat{\xi}$ be the element in $\mathfrak{t}^1$ such that $\xi = \dot{\sigma}(\hat{\xi})$ and $\exp : \mathfrak{t}^1 \to T^1$ the exponential map. Then for $t \in [0, 1]$, $\sigma_t = \tau_{\exp t\hat{\xi}}$.

Let $\alpha \in S_F^p(L)$. Since the flow σ_t is periodic, the map Σ_t constructed in the previous section will have the property that

$$(\Sigma_1 \alpha)_x = l(x)\alpha_x$$

for $x \in M$, where $l(x) \in T^1$ since ∇ is Hermitian. In this way we get a function

$$l : M \to T^1,$$

which is obviously invariant under the flow σ_t. For an arbitrary point $x \in M$, $l(x)$ may not be equal to 1. This means that the lifting of the one parameter flow on the bundle by parallel transport may not be periodic and therefore it may not give a lifting of the action of T^1 on L by bundle transformations. Comparing $\Sigma_1\alpha$ with α, we find that

$$\begin{aligned}\Sigma_1\alpha - \alpha &= \Sigma_1\alpha - \Sigma_0\alpha = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\Sigma_s\alpha\,\mathrm{d}s = \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Sigma_{s+t}\alpha\,\mathrm{d}s \\ &= \int_0^1 \Sigma_s \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Sigma_{s+t}\alpha\,\mathrm{d}s = \int_0^1 \Sigma_s(-\mathcal{L}_\xi\alpha)\,\mathrm{d}s = -\mathcal{L}_\xi \int_0^1 \Sigma_s\alpha\,\mathrm{d}s, \end{aligned} \tag{9}$$

where we have used (7) and (8) and the fact that

$$\mathcal{L}_\xi \circ \Sigma_s = \Sigma_s \circ \mathcal{L}_\xi,$$

which follows directly from the definitions. We define the map

$$I : S_F^p(L) \to S_F^p(L)$$

by setting

$$I\alpha = \int_0^1 \Sigma_s \alpha \, \mathrm{d}s$$

for $\alpha \in S_F^p(L)$. From the definition of Σ_t it is straightforward to check that Σ_t commutes with ∂^F, therefore so does I. Using this property as well as Cartan's identity for the Lie derivative and the definition of the map I, we can rewrite (9) as

$$\begin{aligned}\Sigma_0\alpha - \Sigma_1\alpha &= \mathcal{L}_\xi \circ I\alpha = \partial^F \circ i(\xi) \circ I\alpha + i(\xi) \circ \partial^F \circ I\alpha \\ &= \partial^F \circ i(\xi) \circ I\alpha + i(\xi) \circ I \circ \partial^F \alpha.\end{aligned} \tag{10}$$

We define $J : S_F^p(L) \to S_F^{p-1}(L)$ to be the composition

$$J = i(\xi) \circ I.$$

Since $\Sigma_0\alpha - \Sigma_1\alpha = (1-l)\alpha$, (10) implies that J satisfies the equation

$$(1-l)\alpha = \partial^F \circ J\alpha + J \circ \partial^F \alpha \tag{11}$$

for all $\alpha \in S_F^p(L)$ and $p \geq 1$. For $p = 0$, Cartan's identity reads as

$$\mathcal{L}_\xi \alpha = i(\xi) \circ \partial^F \alpha,$$

therefore (10) becomes

$$(1-l)\alpha = J \circ \partial^F \alpha \tag{12}$$

for $\alpha \in S_F^0(L)$. Now we are able to study in more detail the properties of the function l.

Lemma 3.1 (Rawnsley [8]). $l \in C_F(M)$.

Proof. Let $s \in S_F^0(L)$. Then

$$(1-l)s = J \circ \partial^F s \Rightarrow (1-l)\partial^F s - d^F l \otimes s = \partial^F \circ J \circ \partial^F s.$$

On the other hand, $\partial^F s \in S_F^1(L)$, therefore (13) gives

$$\partial^F \circ J \circ \partial^F s = (1-l)\partial^F s.$$

It follows that $d^F l \otimes s = 0$ for all $s \in S_F^0(L)$, therefore $d^F l = 0$. □

In the case that the curvature of ∇ is the restriction of a symplectic form on M, we shall give in a later section a different proof of this fact.

The set $l^{-1}(1)$ consists of those orbits along which the holonomy of the connection ∇ is trivial. We set $Y := l^{-1}(1)$ and we shall call Y the *Bohr–Sommerfeld set of the flow* σ_t (or of the free action of the circle T^1 on M). In general we do not know what this set looks like. However, the "size" of Y can give information about the space S_F (or $H^0(M, \mathcal{S}_F)$) of global sections of the sheaf $\mathcal{S}_F$.

Lemma 3.2. *Assume that the complement of Y in M is dense in M. Then* $S_F = \{0\}$.

Proof. Let $s \in S_F$. Then $\partial^F s = 0$, therefore $0 = J \circ \partial^F s = (1-l)s$, i.e. $s = 0$ on $M - Y$. Since this set is dense by assumption and s is continuous, s must be identically zero. □

This result does not need Y to be a smooth, or even a topological manifold. If we assume, however, that 1 is a regular value of l then Y is a smooth submanifold of M of codimension one and in this case Y can also give information about the behaviour of the global sections of a vector bundle $E \to M$ near Y. More precisely, we have the following theorem.

Theorem 3.3 (Rawnsley [8]). *Suppose that* 1 *is a regular value of* l. *Then a smooth section* s *of* L *vanishes on* Y *if and only if there exists a smooth section* r *of* L *such that* $s = (1-l)r$.

The proof is analogous to that of the Schwartz lemma for smooth functions, which vanish on the zero set of a function and we refer the reader to Theorem 4 in [8] for a full proof.

From now on we shall always assume that 1 is a regular value of l. Since $l \in C_F(M)$, F is also a subtangent bundle on Y and therefore it is an involutive and strongly integrable sub-bundle of $TY^{\mathbb{C}}$. Let $L|_Y$ be the restriction of L on Y with the restricted connection, denoted again by ∇. Since F is tangent to Y, the restriction of $\mathcal{S}_F$ on Y is isomorphic to the sheaf of germs of local sections of $L|_Y$, which are covariantly constant along F. This will also be denoted by $\mathcal{S}_F$. From the connection ∇ on $L|_Y$, we can construct operators $\partial^F : S_F^p((L|_Y)|_U) \to S_F^{p+1}((L|_Y)|_U)$ for all open sets U of Y and $p \geq 1$. This spaces form presheaves and ∂^F are homomorphisms of presheaves. Therefore, we get an associated complex of sheaves, which is a resolution of $\mathcal{S}_F$ on Y by soft sheaves, therefore by Theorem 2.1, the cohomology groups $H^p(Y, \mathcal{S}_F)$ can be identified with the cohomology groups of the complex of $L|_Y$-valued F-differential forms on Y.

The map $J : S_F^p(L) \to S_F^{p-1}(L)$ does not in general induce a map $\hat{J} : H^p(M, \mathcal{S}_F) \to H^{p-1}(M, \mathcal{S}_F)$, since it does not commute with ∂^F. This obstruction is given by the left-hand side of (11). However, this vanishes on Y, so there is a well defined map

$$\hat{J} : H^p(M, \mathcal{S}_F) \to H^{p-1}(Y, \mathcal{S}_F),$$

given by

$$\hat{J}([\alpha]) - [J\alpha|_Y]$$

for $\alpha \in S_F^p(L)$. The following theorem states a fundamental property of $\hat{J}$ and it is an improved version of Theorem 5 in [8].

Theorem 3.4. *Assume that* $H^p(M, \mathcal{S}_F) = \{0\}$ *for* $0 \leq p < p_0$ *and that* $H^{p_0}(M, \mathcal{S}_F) \neq \{0\}$. *Then* $\hat{J} : H^p(M, \mathcal{S}_F) \to H^{p-1}(Y, \mathcal{S}_F)$ *is an isomorphism for* $1 \leq p \leq p_0$ *and injective for* $p > p_0$.

Proof. We prove first that $\hat{J}$ is an isomorphism for $p = 1$. Let α be a closed form in $S_F^1(L)$ such that $J\alpha|_Y = 0$. Then from Theorem 3.3, we can write $J\alpha = (1-l)s$ for some smooth section s of L and $\partial^F \circ J\alpha = (1-l)\partial^F s$. Therefore $(1-l)\alpha = (1-l)\partial^F s$, which means that $[\alpha] = 0$ and $\hat{J}$ is injective for $p = 1$. Assume now that s is a global section of $L|_Y$ such that $\partial^F s = 0$. Since Y is a closed submanifold of M, we can extend s to a global section s_1 of L. Then $\partial^F s_1|_Y = 0$, therefore by Theorem 3.3 $\partial^F s_1 = (1-l)\alpha$ for some α in $S^1_F(L)$. Then $(1-l)J\alpha = J \circ \partial^F s_1 = (1-l)s_1$, therefore $J\alpha = s_1$ and $J\alpha|_Y = s$. Therefore $\hat{J}$ is surjective for $p = 1$.

Let now α be a closed form in $S^p_F(L)$ for $p \geq 2$ and assume that $\hat{J}[\alpha] = 0$. Then $J\alpha|_Y = \partial^F \beta$ for some β in $S^{p-2}_F(L|_Y)$. Let β_1 be in $S^{p-2}_F(L)$ such that $\beta_1|_Y = \beta$. Then $J\alpha - \partial^F \beta_1$ vanishes on Y, therefore we can write $J\alpha - \partial^F \beta_1 = (1-l)\gamma$ for some γ in $S^{p-1}_F(L)$. Since α is closed (11) gives $(1-l)\alpha = \partial^F J\alpha = (1-l)\partial^F \gamma$, which implies that $\alpha = \partial^F \gamma$, i.e. $[\alpha] = 0$. Therefore $\hat{J}$ is injective.

If $p_0 = 1$ then the theorem is proved. If $p_0 \geq 2$ it remains to prove that $\hat{J}$ is surjective for $1 \leq p \leq p_0$. Suppose $1 \leq p \leq p_0$. Let $[\beta] \in H^{p-1}(Y, \mathcal{S}_F)$, where $\beta \in S^{p-1}_F(L|_Y)$ and satisfies $\partial^F \beta = 0$. We extend β to $\tilde{\beta} \in S^{p-1}_F(L)$. Then

$$(\partial^F \tilde{\beta})|_Y = \partial^F(\tilde{\beta}|_Y) = \partial^F \beta = 0.$$

By Theorem 3.3, $\partial^F \tilde{\beta} = (1-l)\alpha$ for some $\alpha \in S^p_F(L)$. Therefore

$$J \circ \partial^F \tilde{\beta} = J(1-l)\alpha = (1-l)J\alpha.$$

On the other hand, $J \circ \partial^F \tilde{\beta} = (1-l)\tilde{\beta} - \partial^F \circ J\tilde{\beta}$, which implies that $\partial^F \circ J\tilde{\beta}|_Y = 0$, therefore $\partial^F \circ J\tilde{\beta} = (1-l)\tilde{\gamma}$ for some $\tilde{\gamma} \in S^{p-1}_F(L)$. Moreover, $0 = \partial^F \circ \partial^F \circ J\tilde{\beta} = (1-l)\partial^F \tilde{\gamma}$, hence $\tilde{\gamma}$ is ∂^F-closed. Since $H^p(M, \mathcal{S}_F) = \{0\}$ for $1 \leq p \leq p_0$ by assumption, we conclude that either $\tilde{\gamma} = 0$ for $p = 1$ by Lemma 3.2, or by Theorem 2.1 that $\tilde{\gamma}$ is ∂^F-exact if $p > 1$, i.e. $\tilde{\gamma} = \partial^F \tilde{\gamma}_1$ for some $\tilde{\gamma}_1 \in S^{p-2}_F(L)$. In the first case it follows that

$$(1-l)J\alpha = (1-l)\tilde{\beta} \Rightarrow J\alpha = \tilde{\beta} \Rightarrow \hat{J}[\alpha] = [\tilde{\beta}|_Y] = [\beta].$$

In the second case, we have that

$$(1-l)J\alpha = (1-l)\tilde{\beta} - (1-l)\partial^F \tilde{\gamma}_1 \Rightarrow J\alpha = \tilde{\beta} - \partial^F \tilde{\gamma}_1,$$

i.e.,

$$\hat{J}[\alpha] = [(\tilde{\beta} - \partial^F \tilde{\gamma}_1)|_Y] = [\beta - \partial^F(\tilde{\gamma}_1|_Y)] = [\beta].$$

An obvious but quite useful corollary of the above arguments is the following. □

Corollary 3.5. *Let p_1 be the smallest positive integer such that $H^{p_1}(Y, \mathcal{S}_F)$ is not zero. Then $p_0 \geq p_1 + 1$, where p_0 is as in Theorem* 3.4.

The action of T^1 defined by the flow σ_t is proper since T^1 is compact and free by assumption. It also preserves Y, since l is T^1-invariant. Let Z be the quotient Y/T^1 and $\pi : Y \to Z$ the orbit map. Our assumptions guarantee that Z is a smooth manifold and π a smooth submersion (see, for example, Proposition 4.1.23 in [1]). The action τ on M induces an action $\tilde{\tau}$ on *TM* defined by $\tilde{\tau}_g \cdot (x, \eta_x) = (\tau_g \cdot x, \tau_{g_*}\eta_x)$ for $g \in T^1$, $x \in M$, $\eta_x \in T_x M$. By extending this complex linearly, we can define an action on $TM^{\mathbb{C}}$. Since

$\xi \in \Gamma(F)$, this action preserves F, so we can define $\tilde{F}$ as the quotient $F|_Y/T^1$. This is a subtangent bundle on Z. We shall assume that $\tilde{F}$ is involutive and strongly integrable.

In Section 7 of [8] the author describes how the line bundle $p : L \to M$ with the F-connection ∇ induces a line bundle $\tilde{p} : \tilde{L} \to Z$ and an F-connection $\tilde{\nabla}$, such that

$$\pi^*\tilde{L} \simeq L|_Y, \qquad \pi^*\tilde{\nabla} = \nabla|_Y.$$

We recall briefly the construction. We define the equivalence relation $\sim$ on L as follows. Let $p_1, p_2 \in M$, $x_1 = \pi(p_1)$ and $x_2 = \pi(p_2)$. Then $p_1 \sim p_2$ if and only if there exists a $g \in T^1$ such that $x_2 = g \cdot x_1$ and p_2 is obtained by parallel transport along the curve $\gamma : [0, t] \to M$ given by $\gamma(s) = \tau_{\exp(\hat{\xi}s)}(x_1)$, where $\exp(\hat{\xi}t) = g$. Although $L/\sim$ does not define a line bundle over M/T^1, $\tilde{L} := L|_Y/\sim$ can be given a structure of a line bundle over Z with the property $\pi^*\tilde{L} \simeq L|_Y$. This is because we can trivialize the bundle $L|_Y$ over a collection of T^1 invariant open sets $U \subset Y$, which cover Y. Moreover, we can choose local sections s_U of this trivialization to satisfy $\nabla s_U = 0$, which induce local sections $\tilde{s}_V$ of $\tilde{L} \to Z$, where $U = \pi^{-1}(V)$. We can define now a connection $\tilde{\nabla}$ on $\tilde{L}$ by requiring that $\tilde{\nabla}\tilde{s}_V = 0$.

We define now maps $\partial^{\tilde{F}} : S^p_{\tilde{F}}(\tilde{L}) \to S^{p+1}_{\tilde{F}}(\tilde{L})$ for $p \geq 1$. We define $\tilde{S}^p_F(L)$ as the subset of $S^p_F(L|_Y)$ given by the pull-back $\tilde{S}^p_F(L) = \pi^* S^p_{\tilde{F}}(\tilde{L})$. The following proposition gives a more explicit characterization of $\tilde{S}^p_F(L)$.

Proposition 3.6 (Rawnsley [8]). *Let $\alpha \in S^p_F(L|_Y)$. Then $\alpha \in \tilde{S}^p_F(L)$, if and only if $i(\xi)\alpha = 0$ and $\mathcal{L}_\xi\alpha = 0$.*

For a proof, we refer to the proofs of Propositions 3 and 4 in Section 7 of [8]. Using this proposition together with Cartan's identity we can easily deduce the following corollary.

Corollary 3.7 (Rawnsley [8]). $\partial^F \tilde{S}^p_F(L) \subset \tilde{S}^{p+1}_F(L)$.

Therefore, the spaces $\tilde{S}^p_F(L)$, $p = 0, 1, \ldots$, together with the maps ∂^F form a complex

$$0 \to \tilde{S}^p_F(L) \xrightarrow{\partial^F} \tilde{S}^p_F(L) \xrightarrow{\partial^F} \cdots \xrightarrow{\partial^F} \tilde{S}^p_F(L) \xrightarrow{\partial^F} 0,$$

which is called the complex of *basic* forms on Y. Let $\tilde{H}^p(Y, \mathcal{S}_F)$ be the cohomology of this complex. There is a natural map

$$k : \tilde{H}^p(Y, \mathcal{S}_F) \to H^p(Y, \mathcal{S}_F),$$

which maps the class in $\tilde{H}^p(Y, \mathcal{S}_F)$ of a form α in $\tilde{S}^p_F(L)$ to the class of the same form in $H^p(Y, \mathcal{S}_F)$. If $\alpha \in \tilde{S}^p_F(L) \subset S^p_F(L|_Y)$ and $\alpha = \partial^F\gamma$ for some $\gamma \in S^{p-1}_F(L|_Y)$, then γ may not be an element of $\tilde{S}^{p-1}_F(L)$. This implies that neither k need to be injective nor surjective. However, k is an isomorphism in degree 0. Moreover, it is obvious from the definitions that $\tilde{H}^p(Y, \mathcal{S}_F)$ is isomorphic to $H^p(Z, \mathcal{S}_{\tilde{F}})$.

Let $\alpha \in S^p_F(L)$. Then

$$i(\xi)(J\alpha|_Y) = (i(\xi)J\alpha)|_Y = (i(\xi)i(\xi)I\alpha)|_Y = 0.$$

Moreover,

$$\begin{aligned}\mathcal{L}_\xi J\alpha &= \partial^F i(\xi)J\alpha + i(\xi)\partial^F J\alpha \\ &= i(\xi)\partial^F J\alpha = i(\xi)((1-l)\alpha - J\partial^F\alpha) = (1-l)i(\xi)\alpha,\end{aligned}$$

therefore

$$\mathcal{L}_\xi (J\alpha)|_Y = (\mathcal{L}_\xi J\alpha)|_Y = 0.$$

Now Proposition 3.6 implies that $J\alpha|_Y \in \tilde{S}^{p-1}_F(L)$. It follows that we can write $\hat{J}$ as the composition $\hat{J} = k \circ \tilde{J}$, where

$$\tilde{J} : H^p(M, \mathcal{S}_F) \to H^{p-1}(Z, \mathcal{S}_{\tilde{F}}),$$

where for $\alpha \in S^p_{F|_Y}(L)$,

$$\tilde{J}[\alpha] = [(J\alpha)|_Y] \in \tilde{H}^p(Y, \mathcal{S}_F) \simeq H^p(Z, \mathcal{S}_{\tilde{F}}).$$

The map $\tilde{J}$ is injective since $\hat{J}$ is injective. For $p = 1$, since k, $\hat{J}$ are isomorphisms so is $\tilde{J}$. As a conclusion, we have the following theorem proved in [8] (Theorem 6).

Theorem 3.8. *The map $\tilde{J} : H^p(M, \mathcal{S}_F) \to H^{p-1}(Z, \mathcal{S}_{\tilde{F}})$ is an isomorphism for $p = 1$ and injective for $p > 1$.*

Theorem 3.8 suggests that the cohomology of the induced sheaf $\mathcal{S}_{\tilde{F}}$ on the quotient Z contains all the information that the cohomology of $\mathcal{S}_F$ can give.

Remark. An analogue of Theorem 3.4 does not seem to hold for $\tilde{J}$. The reason is that $H^p(Z, \mathcal{S}_{\tilde{F}})$ is not isomorphic to $H^p(Y, \mathcal{S}_F)$, but it is only related to it via the map k, the behaviour of which we do not know in general.

4. Generalization for a k-dimensional torus T^k

In this section, we generalize the theory developed in Section 3 for free circle actions to the case of a free action of a k-dimensional torus T^k on M. Theorems 3.4 and 3.8 generalize Theorems 3.4 and 3.8 of the previous section for free actions of a k-dimensional torus T^k on M.

We assume that M carries all the structure it has been given at the beginning of Section 3 and that we have a smooth and free action of the torus T^k on M. For this action we assume that all its infinitesimal generators are sections of F. We choose k subgroups $T_1, \ldots, T_k$ of T^k, each of which is isomorphic to T^1 and so that the homotopy classes $[T_i] \in \pi_1(T^k)$, $i = 1, \ldots, k$ form an $\mathbb{Z}$-basis of $\pi_1(T^k)$. Then T^k is isomorphic to $T_1 \times \cdots \times T_k$. Choose a corresponding basis $\hat{\xi}_1, \ldots, \hat{\xi}_k$ of the Lie algebra $\mathfrak{t}^k$ of T^k, i.e. $\hat{\xi}_i$ generates T_i, and $\{\hat{\xi}_1, \ldots, \hat{\xi}_k\}$

is an $\mathbb{Z}$-basis for the kernel of the exponential map $\exp : \mathfrak{t}^k \to T^k$. Let $\xi_1, \ldots, \xi_k$ be the infinitesimal generators of the action of T^k corresponding to $\hat{\xi}_1, \ldots, \hat{\xi}_k$, respectively. Since the action is free the flows $\sigma_t^{(i)}$ of ξ_i, $i = 1, \ldots, k$ are periodic with period one. Each of the groups T_i, $i = 1, \ldots, k$ of T^k acts freely on M and the generator of its action is in F, so we can construct a function $l_i : M \to T^1$ as in the previous section, measuring the parallel transport along T_i-orbits.

Lemma 4.1. *The functions l_i, $i = 1, \ldots, k$ are T^k-invariant.*

Proof. By Lemma 3.1, l_i is in $C_F(M)$ for $i = 1, \ldots, k$. Since the infinitesimal generators of the action of T^k are all in F, we have that $\xi_j l_i = 0$ for $i, j = 1, \ldots, k$. Therefore, l_i is invariant under the flow $\sigma_t^{(j)}$ for $i, j = 1, \ldots, k$. Let now $g \in T^k$. We need to show that $l_i(g \cdot x) = l_i(x)$ for all $x \in M$. Since the action is free we can choose $t_1, \ldots, t_k$ such that $g \cdot x = \sigma_{t_1}^{(1)} \circ \cdots \circ \sigma_{t_k}^{(k)}(x)$. Since l_i is invariant under each of $\sigma_t^{(j)}$ it follows that $l_i(g \cdot x) = l_i(x)$. □

We denote by Y_i the Bohr–Sommerfeld set of the action of T_i, $i = 1, \ldots, k$. Assume that 1 is a regular value of l_1. Then Y_1 is a closed submanifold of M of codimension one. We can make all the constructions of the previous section on Y_1 instead of Y. We denote by L_1 be the bundle $L|_{Y_1}$ and for simplicity we still denote by ∇ the connection $\nabla|_{Y_1}$ and $\mathcal{S}_F$ and by $\mathcal{S}_F$ the sheaf $\mathcal{S}_F|_{Y_i}$. Let p_0 be the smallest integer such that $H^{p_0}(M, \mathcal{S}_F) \neq \{0\}$. From Section 3, we know that $p_0 \geq 1$.

Since l_1 is T^k-invariant, T^k acts freely on Y_1 therefore so does each of the subgroups T_j, $j = 1, \ldots, k$ and in particular T_2. Let Y_{12} be the Bohr–Sommerfeld set of the action of T_2^1 on Y_1. Obviously, $Y_{12} = Y_1 \cap Y_2$. We also denote by $\mathcal{S}_F$ the sheaf $\mathcal{S}_F|_{Y_{12}}$. If 1 is a regular value of the map $l_2|_{Y_1}$ then Y_{12} is a closed submanifold of Y_1, and of codimension one. In this case F is also tangent to Y_{12} and we can consider the line bundle $L_{12} \simeq L|_{Y_{12}}$, the F-connection $\nabla|_{Y_{12}}$ and the sheaf $\mathcal{S}_F|_{Y_{12}}$ again denoted by ∇ and $\mathcal{S}_F$, respectively. To keep our notation simple we shall also denote by J the map

$$J : S_F^p(L_1) \to S_F^{p-1}(L_1)$$

constructed in the previous section by first averaging the parallel transport between points on a T_2-orbit and then contracting by ξ_2. This induces a map

$$\hat{J} : H^p(Y_1, \mathcal{S}_F) \to H^{p-1}(Y_{12}, \mathcal{S}_F)$$

for $p \geq 1$. Let p_1 be the smallest integer for which $H^{p_1}(Y_1, \mathcal{S}_F) \neq \{0\}$. By Lemma 3.2, $p_1 \geq 1$ and the map $\hat{J} : H^p(Y_1, \mathcal{S}_F) \to H^{p-1}(Y_{12}, \mathcal{S}_F)$ is an isomorphism for $0 \leq p \leq p_1$ and injective for $p > p_1$. Moreover,

$$H^1(M, \mathcal{S}_F) \simeq H^0(Y_1, \mathcal{S}_F) = \{0\},$$

i.e. $p_0 \geq p_1 + 1 \geq p_2 + 2$. It is obvious now how we can continue. Suppose that $1 \leq m < k$. Let $Y_{1,\ldots,m} = Y_1 \cap \cdots \cap Y_m$. Then F is a strongly integrable subtangent bundle on

$Y_{1,\ldots,m}$. We denote by $L_{1,\ldots,m}$ the restriction of L on $Y_{1,\ldots,m}$ with the induced F-connection and sheaf denoted by ∇ and $\mathcal{S}_F$, respectively. The group T_{m+1} acts freely on $Y_{1,\ldots,m}$ and the Bohr–Sommerfeld set for this action is $Y_{1,\ldots,m+1}$. Assume that 1 is a regular value of $l_{j+1}|_{Y_{1,\ldots,j}}$ for $j=1,\ldots,m$. Then $Y_{1,\ldots,m+1}$ is a closed submanifold of $Y_{1,\ldots,m}$ of codimension one. We denote again by J the map $J : S_F^p(L_{1,\ldots,m}) \to S_F^{p-1}(L_{1,\ldots,m})$ constructed as in the previous section and $\hat{J}$ the corresponding map in cohomology

$$\hat{J} : H^p(Y_{1,\ldots,m}, \mathcal{S}_F) \to H^{p-1}(Y_{1,\ldots m+1}, \mathcal{S}_F)$$

for which Theorem 3.4 says that it is an isomorphism for $p \leq p_{m+1}$ and injective for $p \geq p_{m+1}$, where p_{m+1} is the smallest integer for which the cohomology $H^p(Y_{1,\ldots,m}, \mathcal{S}_F)$ is not trivial. Repeated applications of Lemma 3.2 and Theorem 3.4 give that

$$H^{m-1}(M, \mathcal{S}_F) \simeq H^{m-2}(Y_1, \mathcal{S}_F) \simeq \cdots \simeq H^0(Y_{1,\ldots,m}, \mathcal{S}_F) \simeq \{0\},$$

which shows that

$$p_0 \geq p_1 + 1 \geq p_2 + 2 \geq \cdots \geq p_m + m \geq m. \tag{13}$$

We define the map

$$\hat{J}_{1,\ldots,m} : H^p(M, \mathcal{S}_F) \to H^{p-m}(Y_{1,\ldots,m}, \mathcal{S}_F),$$

to be the composition

$$H^p(M, \mathcal{S}_F) \to H^{p-1}(Y_1, \mathcal{S}_F) \to \cdots \to H^{p-m}(Y_{1,\ldots,m}, \mathcal{S}_F),$$

where all arrows denote the corresponding maps $\hat{J}$. For $m=k$, we get the following theorem as an easy consequence of Theorem 3.4 and Corollary 3.5.

Theorem 4.2. *Assume that* 1 *is a regular value of* l_1 *and of* $l_{r+1}|_{Y_{1,\ldots,r}}$ *for* $r=1,\ldots,k$. *Then the map* $\hat{J} : H^p(M, \mathcal{S}_F) \to H^{p-k}(Y, \mathcal{S}_F)$ *is an isomorphism for* $1 \leq p \leq p_0$ *and injective for* $p > p_0$, *where* $p_0 \geq k$.

Let $Y = Y_1 \cap \cdots \cap Y_k$. We shall call Y the *Bohr–Sommerfeld set of the action of* T^k *on* M. Consider the map

$$l : M \to T^k,$$

given by $l(x) = (l_1(x), \ldots, l_k(x))$ for $x \in M$, where we identify T^k with $T^1 \times \cdots \times T^1$. It is easy to see that $Y = l^{-1}(1)$ and the assumptions of Theorem 4.2 guarantee that $1 \in T^k$ is a regular value of l. In this sense Theorem 4.2 generalizes Theorem 5 in [8] but our assumptions now are stronger than those of Theorem 5 in [8].

We note that we can also give $\hat{J}$ directly in terms of differential forms as follows. For each $p > 0$, let $J_i : S_F^p(L) \to S_F^{p-1}(L)$ be defined as in Section 3 using the action of T_i. We define $J : S_F^p(L) \to S_F^{p-k}(L)$ to be the composition $J = J_1 \circ \cdots \circ J_k$. Let α be a closed differential form in $S_F^p(L)$. We shall show that $J\alpha|_Y$ is also closed. Since $J_k\alpha|_{Y_k}$ is closed in Y_k, the form $(J_{k-1} \circ J_k\alpha)|_{Y_{k-1}\cap Y_k}$, is also closed in $Y_{k-1} \cap Y_k$. By iterating this argument we can finally prove that $J\alpha|_Y$ is closed. Obviously, the induced map $\hat{J} : H^p(M, \mathcal{S}_F) \to H^{p-k}(Y, \mathcal{S}_F)$ is the map $J_{1,\dots,k}$ in Theorem 4.2. We can give the map J explicitly as follows. We denote by $\sigma_{t_1,\dots,t_k}$ the composition $\sigma^{(1)}_{t_1} \circ \cdots \circ \sigma^{(k)}_{t_k}$, where $\sigma^{(i)}_{t_i}$ is the flow of ξ_i for $i = 1, \dots, k$. Let α be a form in $S^p_F(L)$ for $p \geq k$ and let $X_1, \dots, X_{p-k}$ be vectors in F_x for some $x \in M$. Since the vector fields ξ_i are T^k-invariant it easily follows from the definition of J that

$$
\begin{aligned}
&(J\alpha)_x(X_1, \dots, X_{p-k}) \\
&\quad = \int_0^1 \cdots \int_0^1 \alpha(\xi_1, \dots, \xi_{p-k}, (\sigma_{t_1,\dots,t_k})_* X_1, \dots, (\sigma_{t_1,\dots,t_k})_* X_{p-k})\, \mathrm{d}t_1, \dots, \mathrm{d}t_k,
\end{aligned}
$$

where the form α inside the integral is evaluated at $\sigma_{t_1,\dots,t_k}(x)$. This is an integral over the orbit $T^k \cdot x$ and the choice of the basis $\hat{\xi}_1, \dots, \hat{\xi}_k$ for its Lie algebra gives us a choice of a volume form on it. Moreover, as in the case of the circle action, we can prove the following lemma.

Lemma 4.3. *Let α be a form in $S^p_F(L)$ for $p \geq k$. Then the form $J\alpha|_Y$ is basic, in the sense that for all infinitesimal generators ξ, we have $i(\xi)J\alpha|_Y = 0$ and $\mathcal{L}_\xi J\alpha|_Y = 0$.*

Proof. It suffices to show this for the infinitesimal generators $\xi_1, \dots, \xi_1$ since we can write any infinitesimal generator ξ as a linear combination of them. From the definition of J it easily follows that $i(\xi_i)J\alpha|_Y = 0$ for all $i = 1, \dots, k$. Then from Cartan's identity, we get that $\mathcal{L}_{\xi_i} J\alpha = i(\xi_i)\partial^F J\alpha$. Using the equations $\partial^F J_i + J_i \partial^F = (1 - l_i)id$ for $i = 1, \dots, k$, we find that

$$
\partial^F J\alpha + (-1)^{k+1} J \partial^F \alpha = \sum_{i=1}^{k} (-1)^i (1 - l_i)(J_1 \circ \cdots \circ J_{i-1} \circ J_{i+1} \circ \cdots \circ J_k)\alpha. \tag{14}
$$

Since $l_i = 1$ on Y for $i = 1, \dots, k$, we get that $\mathcal{L}_{\xi_i} J\alpha|_Y = (-1)^k i(\xi_i) J\alpha|_Y = 0$. $\square$

Assume that the identity in T^k is a regular value for $l : M \to T^k$. Then Y is a smooth submanifold of M of codimension k. Let $Z = Y/T^k$ and $\tau : Y \to Z$ be the orbit map. This is a smooth submersion. We can complexify the tangent bundle of Z so that the extension of τ_* to $TY^{\mathbb{C}}$ is complex linear. Then the vector bundle $\tilde{F} = \tau_* F$ is a strongly integrable subtangent bundle on Z. We define an equivalence relation $\sim$ on the bundle L by letting two points on the bundle be equivalent if and only if they lie on the same T^k-orbit and we can get one of them by parallel transport of the other along a curve which lies entirely on this orbit. Since the T^k-orbits on Y are absolutely parallel submanifolds we can give the quotient $\tilde{L} = L/\sim$ the structure of a line bundle over Z with a connection $\tilde{\nabla}$ such that $L = \tau^* \tilde{L}$ and $\nabla = \tau^* \tilde{\nabla}$. We do not give more details of this construction since it is completely analogous to that in Section 7 of [8] for circle actions. In precisely the same way one can also show that a form in $S^p_F(L|_Y)$ is the pull-back of a form in $S^p_{\tilde{F}}(\tilde{L})$ if and only if it is basic. This implies that $H^p(Z, \mathcal{S}_{\tilde{F}})$ is isomorphic to the de Rham cohomology of the complex of basic

L-valued differential forms $\tilde{S}^p_F(L|_Y)$. Let us denote this de Rham cohomology of basic forms by $\tilde{H}^p(Y, \mathcal{S}_F)$. As in Section 3, we can define a map $k : \tilde{H}^p(Y, \mathcal{S}_F) \to H^p(Y, \mathcal{S}_F)$, which maps the class of a basic p-form α in $\tilde{H}^p(Y, \mathcal{S}_F)$ to its class in $H^p(Y, \mathcal{S}_F)$. Although we do not know the behaviour of k for an arbitrary degree p, we can easily see that k is the identity map for $p = 0$.

Let α be a closed form in $S^p_F(L)$. By Lemma 4.3, the form $J\alpha|_Y$ is a closed, basic $(p-k)$-form on Y therefore it defines a class in $H^{p-k}(Z, \mathcal{S}_{\tilde{F}})$. This assignment defines a map $\tilde{J} : H^p(M, \mathcal{S}_F) \to H^{p-k}(Z, \mathcal{S}_{\tilde{F}})$ for which we have $\hat{J} = k \circ \tilde{J}$ (where we have identified $H^{p-k}(Z, \mathcal{S}_{\tilde{F}})$ with $\tilde{H}^{p-k}(Y, \mathcal{S}_F)$). Now the fact that k is an isomorphism for $p = 0$ together with Theorem 4.2 imply the following theorem.

Theorem 4.4. *Assume that* $1 \in T^1$ *is a regular value of* l_1 *and of* $l_{m+1}|_{Y_{1,\dots,m}}$ *for* $m = 1, \dots, k-1$. *Then the map* $\tilde{J} : H^p(M, \mathcal{S}_F) \to H^{p-k}(Z, \mathcal{S}_{\tilde{F}})$ *is an isomorphism for* $p = k$ *and injective for* $p > k$. *Moreover*, $H^p(M, \mathcal{S}_F) = \{0\}$ *for* $0 \le p \le k-1$.

5. The symplectic case

In this section, we shall use the results of the previous sections in the context of geometric quantization of a symplectic manifold.

Let (M, ω) be a symplectic manifold and assume that the de Rham class of ω is in the image of the natural map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$ induced by the inclusion of $\mathbb{Z}$ in $\mathbb{R}$. By the theory of prequantization (see [4]), we know that there exists a line bundle $L \to M$ over M, a connection $\tilde{\nabla}$ on L with curvature equal to $-2\pi \mathrm{i}\omega$ and a Hermitian structure h on L compatible with $\tilde{\nabla}$. We fix such *a set of prequantum data* $(L, \tilde{\nabla}, h)$. Let F be a polarization on M. Then F is a Lagrangian, strongly integrable subtangent bundle on M. Considering covariant differentiation with respect to $\tilde{\nabla}$ only at directions in F, we get an F-connection ∇ with curvature equal to $-2\pi \mathrm{i}\omega|_F$. Since F is Lagrangian, ∇ is a flat F-connection on L.

Suppose now that σ is a smooth, free action of the circle T^1 on M. Choose $\hat{\xi} \in \mathfrak{t}^1 \simeq \mathrm{i}\mathbb{R}$ so that $\{n\hat{\xi}, n \in \mathbb{Z}\}$ is the kernel of the exponential map $\exp : \mathfrak{t}^1 \to T^1$. The flow of the corresponding infinitesimal generator ξ has period one and is denoted by σ_t. Since there is a Hermitian structure compatible with ∇, parallel transport along T^1-orbits takes values in T^1. Let $l : M \to T^1$ be the function measuring parallel transport along T^1-orbits, defined as in Section 3. Under our assumptions we can give a geometric proof of Lemma 3.1 as a consequence of the following lemma.

Lemma 5.1. *The function l is in $C^1(M)$ and its derivative is given by*

$$\mathrm{d}l_x = 2\pi \mathrm{i} l \int_0^1 (\sigma_t^* \omega)(\xi_x, \cdot)\, \mathrm{d}t. \tag{15}$$

Proof. Let $x \in M$ and $\gamma : [0, \epsilon] \to M$ a smooth curve, such that $\gamma(0) = x$, $\dot{\gamma}(0) = \eta \in T_x M$. Consider the map $\tilde{S} : [0, \epsilon] \times [0, 1] \to M$ given by $\tilde{S}(s, t) = \exp(t\hat{\xi}) \cdot \gamma(s)$, where $\hat{\xi}$ is an $\mathbb{Z}$-basis for the kernel of the exponential map $\mathfrak{t}^1 \to T^1$. Assuming that η, ξ_x are linearly independent we can choose ϵ small enough, so that $\dot{\gamma}(s), \xi_{\gamma(s)}$ are linearly independent for

$s \in [0, \epsilon]$. Then we can think of $\tilde{S}$ as a parameterization of a smooth surface $S \subset M$ with boundary consisting of the T^1-orbits through $x, \gamma(\epsilon)$. For $r \in (0, \epsilon)$ we denote by $S(r)$ the surface $\tilde{S}([0, r] \times [0, 1])$. From Lemma 1.9.1 of [4], we get that

$$l(\gamma(r)) = l(x)\, \mathrm{e}^{2\pi \mathrm{i} \int_{S(r)} \omega \, \mathrm{d}S}.$$

Using the parameterization $\tilde{S}$, we can write

$$\int_{S(r)} \omega \, \mathrm{d}S = \int_0^r \int_0^1 \omega(\xi, \sigma_{t_*} \dot{\gamma}(s)) \, \mathrm{d}t \, \mathrm{d}s,$$

where, as usual, ξ denotes the infinitesimal generator associated to $\hat{\xi}$. The right-hand side is differentiable with respect to r, therefore so is the $l(\gamma(r))$. In particular, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} l(\gamma(r)) = 2\pi \mathrm{i} l(x) \int_0^1 \omega(\xi_x, \sigma_{t_*}\eta) \, \mathrm{d}t. \tag{16}$$

This formula is also valid in case that ξ_x, η are linearly dependent. In this case the right-hand side is zero because $\dot{\sigma}(\xi)|_{\sigma_t(x)}$ and $\sigma_{t_*}\eta$ are both in $F_{\sigma_t(x)}$ and ∇ is a flat F-connection. If ξ_x, η are linearly dependent, η is also an infinitesimal generator of the action so the left-hand side is zero since l is T^1-invariant. The infinitesimal generator ξ is a T^1-invariant vector field, i.e. it satisfies $\sigma_{t_*}\xi_x = \xi_{\sigma_t(x)}$. Therefore, we can rewrite (16) as

$$\mathrm{d}l_x(\eta) = 2\pi \mathrm{i} l \int_0^1 (\sigma_t^* \omega)_x(\xi_x, \eta) \, \mathrm{d}t \quad \forall \eta \in T_x M. \qquad \Box \tag{17}$$

Corollary 5.2. $l \in C_F(M)$.

Proof. Since ω is smooth and $l \in C^1(M)$, we can differentiate the right-hand side of (15) to conclude that $l \in C^2(M)$. By induction, it follows that $l \in C^\infty(M)$. Since ∇ is a flat F-connection, for $\eta \in F_x$ (17) gives that $\mathrm{d}l_x(\eta) = 0$. Hence, $l \in C_F(M)$. $\Box$

Assume that the action is symplectic. Then (15) becomes

$$\mathrm{d}l = 2\pi \mathrm{i} l \sigma_t{}^* \omega(\xi, \cdot).$$

If the action is almost Hamiltonian, then there exists a linear map $\mu : \mathfrak{t}^1 \to C^\infty(M)$, such that for $\hat{\xi}_1 \in \mathfrak{t}^1$, $\mathrm{d}\mu(\hat{\xi}_1) = \omega(\xi_1, \cdot)$. Then, $\mathrm{d}l = 2\pi \mathrm{i} l \, \mathrm{d}\mu(\hat{\xi})$, which implies that

$$l = \mathrm{e}^{2\pi \mathrm{i}(\mu(\hat{\xi}) + c)} \tag{18}$$

for some $c_1 \in \mathbb{R}$. In fact, our assumption that the infinitesimal generators of the action are sections of F guarantees that if the action is almost Hamiltonian then it is Hamiltonian, i.e. that the image of μ is in our case an Abelian subalgebra of $C^\infty(M)$. Therefore, we can think of $\mu(\hat{\xi})(x) + c$ as the value of a momentum map $\phi : M \to \mathfrak{t}^{1^*}$ evaluated at $\hat{\xi} \in \mathfrak{t}^1$.

Suppose now that the torus T^k acts freely and in a Hamiltonian fashion on (M, ω). Then all values of the momentum map $\phi : M \to (\mathfrak{t}^k)^*$ are regular, since the rank of the derivative of ϕ changes only at points with non-trivial stabilizer groups. We choose a basis $\{\hat{\xi}_1, \dots, \hat{\xi}_k\}$ for the kernel of the exponential map $\exp : \mathfrak{t}^k \to T^k$ and so that the flows of the infinitesimal generators $\xi_1, \dots, \xi_k$ corresponding to $\hat{\xi}_1, \dots, \hat{\xi}_k$ have period one. We can think of the set $\{\hat{\xi}_1, \dots, \hat{\xi}_k\}$ as a basis for the lattice Λ^k of integral elements of $\mathfrak{t}^k$, which is isomorphic to $\pi_1(T^k)$. Each of the basis elements $\hat{\xi}_i$ exponentiates to a circle T_i in T^k and since our connection is Hermitian, parallel transport along T_i-orbits takes values in T^1. We can form the function $l_i : M \to T^1$ as in the previous sections measuring the parallel transport around T_i-orbits. From the discussion above it follows that we can choose a momentum map for the action $\phi : M \to (\mathfrak{t}^k)^*$, so that

$$l_i(x) = \mathrm{e}^{2\pi \mathrm{i}\phi(x)(\hat{\xi}_i)} \quad \forall x \in M, \quad i = 1, \dots, k.$$

We consider now the function $l : M \to T^k$ constructed as in Section 3. This depends on this choice of basis for the lattice Λ^k. The Bohr–Sommerfeld set $Y = l^{-1}(1)$ of the action of T^k on (M, ω) is precisely the set $\phi^{-1}(\Lambda^k) = \cup_{\mu \in \Lambda^k} \phi^{-1}(\mu)$. Moreover, the set $Z = Y/T^k$ is the union of all the reduced spaces at integral levels of ϕ. Since all values of l are regular, the assumptions of Theorems 4.2 and 4.4 are satisfied. Moreover, the theory of Marsden and Weinstein (see [5]) gives that each $Z_\mu := \phi^{-1}(\mu)/T^k$ is a symplectic manifold for all values μ in the image of the momentum map. Moreover, for integral μ the pull-back of the induced line bundle on Z_μ, is the restriction of L on $\phi^{-1}(\mu)$ therefore ω_μ is the curvature of the induced connection on L_μ. This means that the bundle that our construction induces on each of the integral reduced spaces is the prequantization bundle in the theory of Kostant. If F is a polarization of (M, ω) and $\dim M = 2n$, then $\dim F = n$ and since F contains the tangent space of the T^k-orbits which has dimension k, the induced subtangent bundle $\tilde{F}$ on Z has dimension $n - k$. Moreover, $\tilde{F}$ is isotropic since F is isotropic, therefore $\tilde{F}$ is Lagrangian. It follows that $\tilde{F}$ is a polarization on (Z_μ, ω_μ). Therefore, we have the following theorem.

Theorem 5.3. *Assume that the torus T^k acts freely and in a Hamiltonian fashion on the prequantizable symplectic manifold (M, ω) with momentum map $\phi : M \to \mathfrak{t}^{k^*}$. Let F be a strongly integrable polarization on (M, ω) which contains the generators of this action. Then for each integral level set of ϕ, the reduced space inherits naturally (by the orbit map) a polarization $\tilde{F}$ and a prequantization line bundle. Moreover, the maps $\hat{J} : H^p(M, \mathcal{S}_F) \to H^{p-k}(Y, \mathcal{S}_F)$, $\tilde{J} : H^p(M, \mathcal{S}_F) \to H^{p-k}(Y, \mathcal{S}_{\tilde{F}})$ are isomorphisms for $p = k$ and injective for $p > k$, where Y is the union of the integral level sets of ϕ and Z is the union of the corresponding reduced spaces at integral levels of ϕ.*

6. Some applications

It is interesting to compare Theorem 5.3 with Theorem 1 (or Theorem 5.2) in [3]. In [3] the authors consider an action of a compact Lie group G (not necessarily Abelian) and they use a positive polarization which is preserved under the G-action on (M, ω). Theorem 5.3 states an analogous result when the polarization considered contains k real directions which are tangent to the orbits of a torus action. We see that while using a positive polarization it is natural to compare cohomology groups on M with cohomology groups of the same degree on each of the integral reduced spaces, in this paper, it seems more natural to compare cohomology groups of different degrees, via the map $\tilde{J}$.

There is an overlap of our results with those of Śniatycki concerning the geometric quantization of a symplectic manifold using a real polarization F. In [10], Śniatycki defines the Bohr–Sommerfeld set Y as the union of those leaves of F on which ∇ has trivial holonomy. If these leaves of F are spanned by n commuting, complete Hamiltonian vector fields then each of them is diffeomorphic to $T^{p_0} \times \mathbb{R}^{n-p_0}$ for some p_0 (see, for example [1, Theorem 5.2.21]). The main result in [10] is that $H^{p_0}(M, \mathcal{S}_F)$ is isomorphic to $H^0(Y, \mathcal{S}_F)$ and $H^p(M, \mathcal{S}_F) = \{0\}$ for $p \neq p_0$. These results are obtained without the use of an action of the torus of (M, ω). However, assuming that the k-dimensional tori for $k \leq p_0$ contained in the leaves of F are orbits of a free action of the torus T^k on (M, ω), our theory gives the same result for the cohomology groups of degree at most p_0 without assuming that F is totally real. Therefore according to Śniatycki's result, in the case that F is real the number p_0 in Theorem 4.2 is the rank of the fundamental group of the fibres of F. The vanishing of the cohomology groups in degrees greater than p_0 proved by Śniatycki seems to be a special feature of only real polarizations. In this sense our results give a generalization of Śniatycki's theory in the case that F contains real directions which are tangent to the orbits of a free torus action on M. Our discussion, however, does not show that similar conclusions hold in the case that the strongly integrable polarization F has real directions, which are tangent to a k-dimensional foliation of tori, the leaves of which are not necessarily orbits of a torus action.

Another consequence is that if F is real, we can use Śniatycki's vanishing theorem for the cohomology groups of $\mathcal{S}_F$ in degrees greater than k to conclude that under the assumptions of Theorem 4.4 assumptions the induced sheaf on the integral reduced spaces does not have cohomology in degrees greater than zero.

In the special case when $k = n$, i.e. when F is a real polarization the leaves of which are the orbits of an n-dimensional torus acting on M our theory guarantees the vanishing of the cohomology groups at degrees greater than k. This is because the quotient $Z = Y/T^k$ is a discrete set of points and the induced sheaf on Z has cohomology only at degree zero. Therefore by Theorem 4.4, the sheaf $\mathcal{S}_F$ does not have cohomology at degrees greater than k. Moreover, the dimension of $H^k(M, \mathcal{S}_F)$ for $k = n$ is given by the number of points in Z.

Our results depend strongly on the assumption that the action of the torus is free. This assumption excludes a lot of the interesting cases, like Hamiltonian torus actions on compact symplectic manifolds, where we know that we must have more than one orbit types. In a forthcoming paper, we are going to examine the same questions for quasi-free, Hamiltonian actions of a torus T^k on a symplectic manifold.

Acknowledgements

The results of this paper were obtained while the author was working on his Ph.D. Thesis at the Mathematics Institute of the University of Warwick. The author is grateful to

Prof. John Rawnsley for his guidance and the State Scholarship Foundation of Greece for providing the financial support for this research.

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